

Predictions for fermion masses and mixing from a low energy $SU(3)$ flavor symmetry model with a Light Sterile Neutrino

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Abstract

I report low energy results on the study of fermion masses and mixing for quarks and leptons, including neutrinos within a $SU(3)$ flavor symmetry model, where ordinary heavy fermions, top and bottom quarks and tau lepton become massive at tree level from **Dirac See-saw** mechanisms implemented by the introduction a new set of $SU(2)_L$ weak singlet vector-like fermions U, D, E, N , with N a sterile neutrino. Light fermions obtain masses from one loop radiative corrections mediated by the massive $SU(3)$ gauge bosons. Recent results shows the existence of a low energy space parameter where this model is able to accommodate the known spectrum of quark masses and mixing in a 4×4 non-unitary V_{CKM} as well as the charged lepton masses. Motivated by the recent LSND and MiniBooNe short-baseline neutrino oscillation experiments we fit for the 3+1 scenario the neutrino squared mass differences $m_2^2 - m_1^2 \approx 7.6 \times 10^{-5} \text{ eV}^2$, $m_3^2 - m_2^2 \approx 2.43 \times 10^{-3} \text{ eV}^2$ and $m_4^2 - m_1^2 \approx 0.29 \text{ eV}^2$. The model predicts the D vector like quark mass in the range $M_D = (350 - 900) \text{ GeV}$ and horizontal gauge boson masses of few TeV. These low energy predictions are within LHC possibilities. Furthermore, the above scenario enable us to suppress simultaneously the tree level $\Delta F = 2$ processes for $K^o - \bar{K}^o$ and $D^o - \bar{D}^o$ meson mixing mediated by these extra horizontal gauge bosons within current experimental bounds.

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I. INTRODUCTION

The strong hierarchy of quark and charged lepton masses and quark mixing have suggested to many model building theorists that light fermion masses could be generated from radiative corrections[1], while those of the top and bottom quarks as well as that of the tau lepton are generated at tree level. This may be understood as a consequence of the breaking of a symmetry among families (a horizontal symmetry). This symmetry may be discrete [2], or continuous, [3]. The radiative generation of the light fermions may be mediated by scalar particles as it is proposed, for instance, in references [4, 5] and this author in [15], or also through vectorial bosons as it happens for instance in "Dynamical Symmetry Breaking" (DSB) and theories like " Extended Technicolor " [6].

In this article I address the problem of fermion masses and quark mixing within an extension of the SM introduced by the author[7] which includes a $SU(3)$ gauged flavor symmetry commuting with the SM group. In previous reports[8] we showed that this model has the ingredients to accommodate a realistic spectrum of charged fermion masses and quark mixing. We introduce a hierarchical mass generation mechanism in which the light fermions obtain masses through one loop radiative corrections, mediated by the massive bosons associated to the $SU(3)$ family symmetry that is spontaneously broken, while the masses for the top and bottom quarks as well as for the tau lepton, are generated at tree level by the implementation of "Dirac See-saw" mechanisms implemented by the introduction of a new generation of $SU(2)_L$ weak singlets vector-like fermions. Recently, some authors have pointed out interesting features regarding the possibility of the existence of a sequential fourth generation[9]. Theories and models with extra matter may also provide interesting scenarios for present cosmological problems, such as candidates for the nature of the Dark Matter ([10],[11]). This is the case of an extra generation of vector-like matter, both from theory and current experiments[12]. Due to the fact that the vector-like quarks do not couple to the W boson, the mixing of U and D vector-like quarks with the SM quarks yield an extended 4×4 non-unitary CKM quark mixing matrix. It has pointed out for some authors that these type of vector-like fermions are weakly constrained from Electroweak Precision Data (EWPD) because they do not break directly the custodial symmetry, then main experimental constraints on vector-like matter come from the direct production bounds and their implications on flavor physics. See ref. [12] for further details on constraints for

$SU(2)_L$ singlet vector-like fermions.

Motivated by recent results from the LSND and MiniBooNe short-baseline neutrino oscillation experiments many authors are paying special attention to the study of light sterile neutrinos in the eV-scale to explain the tension in the interpretation of these data[13].

Here we report updated low energy results which accounts for the known quark and charged lepton masses and the quark mixing in a non-unitary $(V_{CKM})_{4\times 4}$. We also include a fit for neutrino masses within a "Dirac See-saw" mechanism with a light sterile neutrino of $m_4 \approx 0.54$ eV.

II. MODEL WITH $SU(3)$ FLAVOR SYMMETRY

A. Fermion content

We define the gauge group symmetry $G \equiv SU(3) \otimes G_{SM}$, where $SU(3)$ is a flavor symmetry among families and $G_{SM} \equiv SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ is the "Standard Model" gauge group of elementary particles. The content of fermions assumes the ordinary quarks and leptons assigned under G as: $\psi_q^o = (3, 3, 2, \frac{1}{3})_L$, $\psi_l^o = (3, 1, 2, -1)_L$, $\psi_u^o = (3, 3, 1, \frac{4}{3})_R$, $\psi_d^o = (3, 3, 1, -\frac{2}{3})_R$, $\psi_e^o = (3, 1, 1, -2)_R$, where the last entry corresponds to the hypercharge Y , and the electric charge is defined by $Q = T_{3L} + \frac{1}{2}Y$. The model also includes two types of extra fermions: Right handed neutrinos $\Psi_\nu^o = (3, 1, 1, 0)_R$, and the $SU(2)_L$ singlet vector-like fermions

$$U_{L,R}^o = (1, 3, 1, \frac{4}{3}) \quad , \quad D_{L,R}^o = (1, 3, 1, -\frac{2}{3}) \quad (1)$$

$$N_{L,R}^o = (1, 1, 1, 0) \quad , \quad E_{L,R}^o = (1, 1, 1, -2) \quad (2)$$

The above fermion content and its assignment under the group G make the model anomaly free. After the definition of the gauge symmetry group and the assignment of the ordinary fermions in the canonical form under the standard model group and in the fundamental 3-representation under the $SU(3)$ family symmetry, the introduction of the right-handed neutrinos is required to cancel anomalies[14]. The $SU(2)_L$ weak singlets vector-like fermions have been introduced to give masses at tree level only to the third family of known fermions through Dirac See-saw mechanisms. These vector like fermions play a crucial role to im-

plement a hierarchical spectrum for quarks and charged lepton masses together with the radiative corrections.

III. SPONTANEOUS SYMMETRY BREAKING

The "Spontaneous Symmetry Breaking" (SSB) is proposed to be achieved in the form:

$$G \xrightarrow{\Lambda_1} SU(2) \otimes G_{SM} \xrightarrow{\Lambda_2} G_{SM} \xrightarrow{\Lambda_3} SU(3)_C \otimes U(1)_Q \quad (3)$$

Here Λ_1 , Λ_2 and Λ_3 are the scales of SSB in order the model to have the possibility to be consistent with the known low energy physics.

A. Electroweak symmetry breaking

To achieve the spontaneous breaking of the electroweak symmetry to $U(1)_Q$, we introduce the scalars: $\Phi = (3, 1, 2, -1)$ and $\Phi' = (3, 1, 2, +1)$, with the VEVs: $\langle \Phi \rangle^T = (\langle \Phi_1 \rangle, \langle \Phi_2 \rangle, \langle \Phi_3 \rangle)$, $\langle \Phi' \rangle^T = (\langle \Phi'_1 \rangle, \langle \Phi'_2 \rangle, \langle \Phi'_3 \rangle)$, where T means transpose, and

$$\langle \Phi_i \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} v_i \\ 0 \end{pmatrix} \quad , \quad \langle \Phi'_i \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ V_i \end{pmatrix} . \quad (4)$$

Assuming $(v_1, v_2, v_3) \neq (V_1, V_2, V_3)$ with $v_1^2 + v_2^2 + v_3^2 = V_1^2 + V_2^2 + V_3^2$, the contributions from $\langle \Phi \rangle$ and $\langle \Phi' \rangle$ yield the W gauge boson mass $\frac{1}{2}g^2(v_1^2 + v_2^2 + v_3^2)W^+W^-$. Hence, if we define as usual $M_W = \frac{1}{2}gv$, we may write $v = \sqrt{2}\sqrt{v_1^2 + v_2^2 + v_3^2} \approx 246$ GeV.

Let me emphasize here that solutions for fermion masses and mixing reported in section V suggest that the dominant contribution to Electroweak Symmetry Breaking comes from the weak doublets which couple to the third family.

B. $SU(3)$ flavor symmetry breaking

To implement a hierarchical spectrum for charged fermion masses, and simultaneously to achieve the SSB of $SU(3)$, we introduce the scalar fields: η_i , $i = 1, 2, 3$, transforming under the gauge group as $(3, 1, 1, 0)$ and taking the "Vacuum Expectation Values" (VEV's):

$$\langle \eta_3 \rangle^T = (0, 0, \mathcal{V}_3) \quad , \quad \langle \eta_2 \rangle^T = (0, \mathcal{V}_2, 0) \quad , \quad \langle \eta_1 \rangle^T = (\mathcal{V}_1, 0, 0) . \quad (5)$$

The above scalar fields and VEV's break completely the $SU(3)$ flavor symmetry. The corresponding $SU(3)$ gauge bosons are defined in Eq.(12) through their couplings to fermions. To simplify computations, we impose a $SU(2)$ global symmetry in the gauge boson masses. So, we assume $\mathcal{V}_1 = \mathcal{V}_2 \equiv \mathcal{V}$ in order to cancel mixing between Z_1 and Z_2 horizontal gauge bosons. Thus, a natural hierarchy among the VEVs consistent with the proposed sequence of SSB in Eq.(3) is $\mathcal{V}_3 \gg \mathcal{V} \gg \sqrt{v_1^2 + v_2^2 + v_3^2} = \frac{v}{\sqrt{2}} \simeq \frac{246 \text{ GeV}}{\sqrt{2}} \simeq 173.9 \text{ GeV} \approx m_t$. Hence, neglecting tiny contributions from electroweak symmetry breaking, we obtain for the gauge bosons masses

$$g_H^2 \left\{ \frac{1}{2} (\mathcal{V})^2 [Z_1^2 + (Y_1^1)^2 + (Y_1^2)^2] + \frac{1}{6} [2(\mathcal{V}_3)^2 + (\mathcal{V})^2] Z_2^2 + \frac{1}{4} ((\mathcal{V}_3)^2 + (\mathcal{V})^2) [(Y_2^1)^2 + (Y_2^2)^2 + (Y_3^1)^2 + (Y_3^2)^2] \right\} \quad (6)$$

Them, we may define the horizontal boson masses

$$\begin{aligned} (M_{Z_1})^2 &= (M_{Y_1^1})^2 = (M_{Y_1^2})^2 = M_1^2 \equiv g_H^2 \mathcal{V}^2 , \\ (M_{Y_2^1})^2 &= (M_{Y_2^2})^2 = (M_{Y_3^1})^2 = (M_{Y_3^2})^2 = M_2^2 \equiv \frac{g_H^2}{2} (\mathcal{V}_3^2 + \mathcal{V}^2) , \\ (M_{Z_2})^2 &= 4/3 M_2^2 - 1/3 M_1^2 \end{aligned} \quad (7)$$

with the hierarchy $M_{Z_2} \gtrsim M_2 > M_1 \gg M_W$. It is worth to emphasize that this $SU(2)$ global symmetry together with the hierarchy of scales in the SSB yield a spectrum of $SU(3)$ gauge boson masses without mixing in quite good approximation. Actually this global $SU(2)$ symmetry plays the role of a custodial symmetry to suppress properly the tree level $\Delta F = 2$ processes mediated by the M_1 lower scale Z_1, Y_1^1, Y_1^2 horizontal gauge bosons.

IV. FERMION MASSES

A. Dirac See-saw mechanisms

Now we describe briefly the procedure to get the masses for fermions. The analysis is presented explicitly for the charged lepton sector, with a completely analogous procedure

for the u and d quarks and Dirac neutrinos. With the fields of particles introduced in the model, we may write the gauge invariant Yukawa couplings, as

$$h \bar{\psi}_l^o \Phi' E_R^o + h_1 \bar{\psi}_e^o \eta_1 E_L^o + h_2 \bar{\psi}_e^o \eta_2 E_L^o + h_3 \bar{\psi}_e^o \eta_3 E_L^o + M \bar{E}_L^o E_R^o + h.c \quad (8)$$

where M is a free mass parameter (because its mass term is gauge invariant) and h , h_1 , h_2 and h_3 are Yukawa coupling constants. When the involved scalar fields acquire VEV's we get, in the gauge basis $\psi_{L,R}^o = (e^o, \mu^o, \tau^o, E^o)_{L,R}$, the mass terms $\bar{\psi}_L^o \mathcal{M}^o \psi_R^o + h.c$, where

$$\mathcal{M}^o = \begin{pmatrix} 0 & 0 & 0 & h v_1 \\ 0 & 0 & 0 & h v_2 \\ 0 & 0 & 0 & h v_3 \\ h_1 \mathcal{V} & h_2 \mathcal{V} & h_3 \mathcal{V}_3 & M \end{pmatrix} \equiv \begin{pmatrix} 0 & 0 & 0 & a_1 \\ 0 & 0 & 0 & a_2 \\ 0 & 0 & 0 & a_3 \\ b_1 & b_2 & b_3 & c \end{pmatrix}. \quad (9)$$

Notice that \mathcal{M}^o has the same structure of a See-saw mass matrix, here for Dirac fermion masses. So, we call \mathcal{M}^o a **"Dirac See-saw"** mass matrix. \mathcal{M}^o is diagonalized by applying a biunitary transformation $\psi_{L,R}^o = V_{L,R}^o \chi_{L,R}$. The orthogonal matrices V_L^o and V_R^o are obtained explicitly in the Appendix A. From V_L^o and V_R^o , and using the relationships defined in this Appendix, one computes

$$V_L^{oT} \mathcal{M}^o V_R^o = \text{Diag}(0, 0, -\sqrt{\lambda_-}, \sqrt{\lambda_+}) \quad (10)$$

$$V_L^{oT} \mathcal{M}^o \mathcal{M}^{oT} V_L^o = V_R^{oT} \mathcal{M}^{oT} \mathcal{M}^o V_R^o = \text{Diag}(0, 0, \lambda_-, \lambda_+). \quad (11)$$

where λ_- and λ_+ are the nonzero eigenvalues defined in Eqs.(51-52), $\sqrt{\lambda_+}$ being the fourth heavy fermion mass, and $\sqrt{\lambda_-}$ of the order of the top, bottom and tau mass for u , d and e fermions, respectively. We see from Eqs.(10,11) that at tree level the See-saw mechanism yields two massless eigenvalues associated to the light fermions:

B. One loop contribution to fermion masses

Subsequently, the masses for the light fermions arise through one loop radiative corrections. After the breakdown of the electroweak symmetry we can construct the generic one

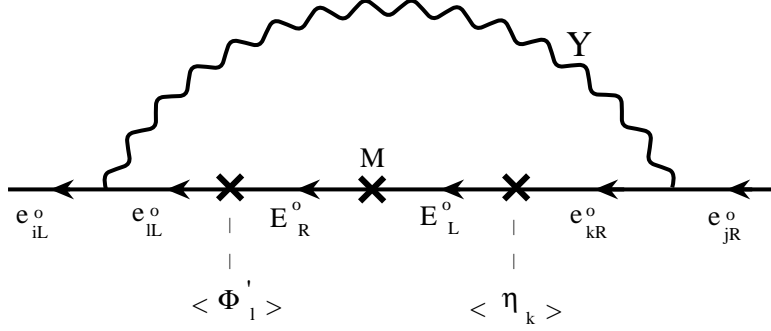


FIG. 1: Generic one loop diagram contribution to the mass term $m_{ij} \bar{e}_{iL}^o e_{jR}^o$

loop mass diagram of Fig. 1. The vertices in this diagram read from the $SU(3)$ flavor symmetry interaction Lagrangian

$$\begin{aligned}
i\mathcal{L}_{int} = \frac{g_H}{2} \Big\{ & (\bar{e}^o \gamma_\mu e^o - \bar{\mu}^o \gamma_\mu \mu^o) Z_1^\mu + \frac{1}{\sqrt{3}} (\bar{e}^o \gamma_\mu e^o + \bar{\mu}^o \gamma_\mu \mu^o - 2\bar{\tau}^o \gamma_\mu \tau^o) Z_2^\mu \\
& + (\bar{e}^o \gamma_\mu \mu^o + \bar{\mu}^o \gamma_\mu e^o) Y_1^{1\mu} + (-i\bar{e}^o \gamma_\mu \mu^o + i\bar{\mu}^o \gamma_\mu e^o) Y_1^{2\mu} \\
& + (\bar{e}^o \gamma_\mu \tau^o + \bar{\tau}^o \gamma_\mu e^o) Y_2^{1\mu} + (-i\bar{e}^o \gamma_\mu \tau^o + i\bar{\tau}^o \gamma_\mu e^o) Y_2^{2\mu} \\
& + (\bar{\mu}^o \gamma_\mu \tau^o + \bar{\tau}^o \gamma_\mu \mu^o) Y_3^{1\mu} + (-i\bar{\mu}^o \gamma_\mu \tau^o + i\bar{\tau}^o \gamma_\mu \mu^o) Y_3^{2\mu} \Big\} , \quad (12)
\end{aligned}$$

where g_H is the $SU(3)$ coupling constant, Z_1 , Z_2 and Y_i^j , $i = 1, 2, 3$, $j = 1, 2$ are the eight gauge bosons. The crosses in the internal fermion line mean tree level mixing, and the mass M generated by the Yukawa couplings in Eq.(8) after the scalar fields get VEV's. The one loop diagram of Fig. 1 gives the generic contribution to the mass term $m_{ij} \bar{e}_{iL}^o e_{jR}^o$

$$c_Y \frac{\alpha_H}{\pi} \sum_{k=3,4} m_k^o (V_L^o)_{ik} (V_R^o)_{jk} f(M_Y, m_k^o) \quad , \quad \alpha_H \equiv \frac{g_H^2}{4\pi} \quad (13)$$

where M_Y is the gauge boson mass, c_Y is a factor coupling constant, Eq.(12), $m_3^o = -\sqrt{\lambda_-}$ and $m_4^o = \sqrt{\lambda_+}$ are the See-saw mass eigenvalues, Eq.(10), and $f(x, y) = \frac{x^2}{x^2 - y^2} \ln \frac{x^2}{y^2}$. Using the results of Appendix A, we compute

$$\sum_{k=3,4} m_k^o (V_L^o)_{ik} (V_R^o)_{jk} f(M_Y, m_k^o) = \frac{a_i b_j M}{\lambda_+ - \lambda_-} F(M_Y, \sqrt{\lambda_-}, \sqrt{\lambda_+}) , \quad (14)$$

$i, j = 1, 2, 3$ and $F(M_Y, \sqrt{\lambda_-}, \sqrt{\lambda_+}) \equiv \frac{M_Y^2}{M_Y^2 - \lambda_+} \ln \frac{M_Y^2}{\lambda_+} - \frac{M_Y^2}{M_Y^2 - \lambda_-} \ln \frac{M_Y^2}{\lambda_-}$. Adding up all the one loop $SU(3)$ gauge boson contributions, we get the mass terms $\bar{\psi}_L^o \mathcal{M}_1^o \psi_R^o + h.c.$,

$$\mathcal{M}_1^o = \begin{pmatrix} R_{11} & R_{12} & R_{13} & 0 \\ R_{21} & R_{22} & R_{23} & 0 \\ R_{31} & R_{32} & R_{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \frac{\alpha_H}{\pi}, \quad (15)$$

$$R_{11} = -\frac{1}{4}F_1(m_{11} + 2m_{22}) - \frac{1}{12}F_{Z_2}m_{11} + \frac{1}{2}F_2m_{33},$$

$$R_{22} = -\frac{1}{4}F_1(2m_{11} + m_{22}) - \frac{1}{12}F_{Z_2}m_{22} + \frac{1}{2}F_2m_{33},$$

$$R_{12} = (\frac{1}{4}F_1 - \frac{1}{12}F_{Z_2})m_{12} \quad , \quad R_{21} = (\frac{1}{4}F_1 - \frac{1}{12}F_{Z_2})m_{21}, \quad (16)$$

$$R_{33} = \frac{1}{3}F_{Z_2}m_{33} - \frac{1}{2}F_2(m_{11} + m_{22}) \quad , \quad R_{13} = -\frac{1}{6}F_{Z_2}m_{13},$$

$$R_{31} = \frac{1}{6}F_{Z_2}m_{31} \quad , \quad R_{23} = -\frac{1}{6}F_{Z_2}m_{23} \quad , \quad R_{32} = \frac{1}{6}F_{Z_2}m_{32}.$$

Here, $F_{1,2} \equiv F(M_{1,2}, \sqrt{\lambda_-}, \sqrt{\lambda_+})$ and $F_{Z_2} \equiv F(M_{Z_2}, \sqrt{\lambda_-}, \sqrt{\lambda_+})$, with M_1 , M_2 and M_{Z_2} the horizontal boson masses, Eq.(7),

$$m_{ij} = \frac{a_i b_j M}{\lambda_+ - \lambda_-} = \frac{a_i b_j}{a b} \sqrt{\lambda_-} c_\alpha c_\beta, \quad (17)$$

and $c_\alpha \equiv \cos \alpha$, $c_\beta \equiv \cos \beta$, $s_\alpha \equiv \sin \alpha$, $s_\beta \equiv \sin \beta$, as defined in the Appendix, Eq.(53).

Therefore, up to one loop corrections we obtain the fermion masses

$$\bar{\psi}_L^o \mathcal{M}^o \psi_R^o + \bar{\psi}_L^o \mathcal{M}_1^o \psi_R^o = \bar{\chi}_L \mathcal{M} \chi_R, \quad (18)$$

with $\mathcal{M} \equiv \left[\text{Diag}(0, 0, -\sqrt{\lambda_-}, \sqrt{\lambda_+}) + V_L^{oT} \mathcal{M}_1^o V_R^o \right]$.

Using V_L^o , V_R^o in Eqs.(47-48) we get the mass matrix in Version I:

$$\mathcal{M} = \begin{pmatrix} m_{11} & m_{12} & c_\beta m_{13} & s_\beta m_{13} \\ m_{21} & m_{22} & c_\beta m_{23} & s_\beta m_{23} \\ c_\alpha m_{31} & c_\alpha m_{32} & (-\sqrt{\lambda_-} + c_\alpha c_\beta m_{33}) & c_\alpha s_\beta m_{33} \\ s_\alpha m_{31} & s_\alpha m_{32} & s_\alpha c_\beta m_{33} & (\sqrt{\lambda_+} + s_\alpha s_\beta m_{33}) \end{pmatrix}, \quad (19)$$

where the mass entries m_{ij} ; $i, j = 1, 2, 3$ are written as:

$$\begin{aligned} m_{11} &= \frac{\eta_+}{a'b'} c_1 H, & m_{12} &= -\frac{\eta_-}{a'b'} \frac{b_3}{b} c_1 H, \\ m_{21} &= \frac{\eta_-}{a'b'} \frac{a_3}{a} c_1 H, & m_{22} &= c_2 \left[\frac{\eta_+}{a'b'} H + \frac{a'b'}{a_3 b_3} (J + \frac{\Delta}{2}) \right], \\ m_{31} &= \frac{\eta_-}{a'b'} \frac{a'}{a} c_1 H, & m_{32} &= c_2 \left[\frac{a'}{a_3} \left(\frac{\eta_+}{a'b'} H + \frac{1}{2} \frac{a'b'}{a_3 b_3} \Delta \right) - \frac{b'}{b_3} J \right], \\ m_{13} &= -\frac{\eta_-}{a'b'} \frac{b'}{b} c_1 H, & m_{23} &= \left[\frac{b'}{b_3} \left(\frac{\eta_+}{a'b'} H + \frac{1}{2} \frac{a'b'}{a_3 b_3} \Delta \right) - \frac{a'}{a_3} J \right], \\ m_{33} &= c_2 \left(\frac{\eta_+}{a_3 b_3} H + J + \frac{1}{6} \frac{a'^2 b'^2}{a_3^2 b_3^2} \Delta - \frac{1}{3} \left[\frac{a'^2 b'^2}{a_3^2 b_3^2} F_1 + \left(1 + \frac{a'^2}{a_3^2} + \frac{b'^2}{b_3^2} \right) F_{Z_2} \right] \right), \end{aligned} \quad (20)$$

For V_L^o , V_R^o of Eqs.(49-50) we get the Version II:

$$\mathcal{M} = \begin{pmatrix} M_{11} & M_{12} & c_\beta M_{13} & s_\beta M_{13} \\ M_{21} & M_{22} & c_\beta M_{23} & s_\beta M_{23} \\ c_\alpha M_{31} & c_\alpha M_{32} & (-\sqrt{\lambda_-} + c_\alpha c_\beta M_{33}) & c_\alpha s_\beta M_{33} \\ s_\alpha M_{31} & s_\alpha M_{32} & s_\alpha c_\beta M_{33} & (\sqrt{\lambda_+} + s_\alpha s_\beta M_{33}) \end{pmatrix}, \quad (21)$$

where the mass terms M_{ij} ; $i, j = 1, 2, 3$ may be obtained from those of m_{ij} as follows

$$M_{11} = m_{22}, \quad M_{12} = -m_{21}, \quad M_{13} = m_{23}$$

$$M_{21} = -m_{12}, \quad M_{22} = m_{11}, \quad M_{23} = -m_{13} \quad (22)$$

$$M_{31} = m_{32}, \quad M_{32} = -m_{31}, \quad M_{33} = m_{33}$$

$$\eta_- = a_1 b_2 - a_2 b_1, \quad \eta_+ = a_1 b_1 + a_2 b_2, \quad \eta_-^2 + \eta_+^2 = a'^2 b'^2 \quad (23)$$

$$a' = \sqrt{a_1^2 + a_2^2}, \quad b' = \sqrt{b_1^2 + b_2^2}, \quad a = \sqrt{a'^2 + a_3^2}, \quad b = \sqrt{b'^2 + b_3^2}, \quad (24)$$

$$c_1 = \frac{1}{2} c_\alpha c_\beta \frac{a_3 b_3}{a b} \frac{\alpha_H}{\pi}, \quad c_2 = \frac{a_3 b_3}{a b} c_1, \quad (25)$$

$$H = F_2 + \frac{\eta_+}{a_3 b_3} F_1, \quad J = F_{Z_2} + \frac{\eta_+}{a_3 b_3} F_2, \quad \Delta = F_{Z_2} - F_1. \quad (26)$$

The diagonalization of \mathcal{M} , Eq.(19) or Eq.(21), gives the physical masses for u, d, e and ν fermions. Using a new biunitary transformation $\chi_{L,R} = V_{L,R}^{(1)} \Psi_{L,R}$; $\bar{\chi}_L \mathcal{M} \chi_R = \bar{\Psi}_L V_L^{(1)T} \mathcal{M} V_R^{(1)} \Psi_R$, with $\Psi_{L,R}^T = (f_1, f_2, f_3, F)_{L,R}$ the mass eigenfields, that is

$$V_L^{(1)T} \mathcal{M} \mathcal{M}^T V_L^{(1)} = V_R^{(1)T} \mathcal{M}^T \mathcal{M} V_R^{(1)} = \text{Diag}(m_1^2, m_2^2, m_3^2, M_F^2), \quad (27)$$

$m_1^2 = m_e^2$, $m_2^2 = m_\mu^2$, $m_3^2 = m_\tau^2$ and $M_F^2 = M_E^2$ for charged leptons. Therefore, the transformation from massless to mass fermions eigenfields in this scenario reads

$$\psi_L^o = V_L^o V_L^{(1)} \Psi_L \quad \text{and} \quad \psi_R^o = V_R^o V_R^{(1)} \Psi_R \quad (28)$$

C. Quark Mixing and non-unitary $(V_{CKM})_{4 \times 4}$

Recall that vector like quarks, Eq.(1), are $SU(2)_L$ weak singlets, and then they do not couple to W boson in the interaction basis. So, the interaction of quarks $f_{uL}^o{}^T = (u^o, c^o, t^o)_L$ and $f_{dL}^o{}^T = (d^o, s^o, b^o)_L$ to the W charged gauge boson is

$$\bar{f}_{uL}^o \gamma_\mu f_{dL}^o W^{+\mu} = \bar{\Psi}_{uL} V_{uL}^{(1)T} [(V_{uL}^o)_{3 \times 4}]^T (V_{dL}^o)_{3 \times 4} V_{dL}^{(1)} \gamma_\mu \Psi_{dL} W^{+\mu}, \quad (29)$$

hence, the non-unitary V_{CKM} of dimension 4×4 is identified as

$$(V_{CKM})_{4 \times 4} \equiv V_{uL}^{(1)T} [(V_{uL}^o)_{3 \times 4}]^T (V_{dL}^o)_{3 \times 4} V_{dL}^{(1)} . \quad (30)$$

For u-quarks in version I and d-quarks in version II,

$$V^o \equiv [(V_{uL}^o)_{3 \times 4}]^T (V_{dL}^o)_{3 \times 4} = \begin{pmatrix} \frac{s_o}{\sqrt{1+r_d^2}} & -c_o & \frac{c_\alpha^d s_o r_d}{\sqrt{1+r_d^2}} & \frac{s_\alpha^d s_o r_d}{\sqrt{1+r_d^2}} \\ \Omega_{11} & \frac{s_o}{\sqrt{1+r_u^2}} & c_\alpha^d \Omega_{13} & s_\alpha^d \Omega_{13} \\ c_\alpha^u \Omega_{31} & \frac{c_\alpha^u s_o r_u}{\sqrt{1+r_u^2}} & c_\alpha^u c_\alpha^d \Omega_{33} & c_\alpha^u s_\alpha^d \Omega_{33} \\ s_\alpha^u \Omega_{31} & \frac{s_\alpha^u s_o r_u}{\sqrt{1+r_u^2}} & s_\alpha^u c_\alpha^d \Omega_{33} & s_\alpha^u s_\alpha^d \Omega_{33} \end{pmatrix} , \quad (31)$$

$$\Omega_{11} = \frac{r_u r_d + c_o}{\sqrt{(1+r_u^2)(1+r_d^2)}} , \quad \Omega_{13} = \frac{r_d c_o - r_u}{\sqrt{(1+r_u^2)(1+r_d^2)}} \quad (32)$$

$$\Omega_{31} = \frac{r_u c_o - r_d}{\sqrt{(1+r_u^2)(1+r_d^2)}} , \quad \Omega_{33} = \frac{r_u r_d c_o + 1}{\sqrt{(1+r_u^2)(1+r_d^2)}} \quad (33)$$

$$s_o = \frac{v_2}{v'} \frac{V_1}{V'} - \frac{v_1}{v'} \frac{V_2}{V'} , \quad c_o = \frac{v_1}{v'} \frac{V_1}{V'} + \frac{v_2}{v'} \frac{V_2}{V'} \quad (34)$$

$$c_o^2 + s_o^2 = 1 , \quad r_u = \left(\frac{a'}{a_3}\right)_u , \quad r_d = \left(\frac{a'}{a_3}\right)_d \quad (35)$$

$V_i, v_i, i = 1, 2$ are related to (e,d) and (u, ν) fermions respectively.

V. NUMERICAL RESULTS

Using the strong hierarchy for quarks and charged leptons masses and the results in[15], we report here the magnitudes of quark masses and mixing coming from the analysis of a low energy parameter space in this model. For this numerical analysis we used the input global parameters $\frac{\alpha_H}{\pi} = 0.2$, $M_1 = 4$ TeV and $M_2 = 1700$ TeV.

A. Sector d:

Parameter space: $(\sqrt{\lambda_-})_d = 4.98$ GeV, $(\sqrt{\lambda_+})_d = 500$ GeV, $r_d = 0.052$, $(\eta_+/a_3b_3)_d = -0.49$, $(\eta_-/\eta_+)_d = 1.3$, $s_\alpha^d = 0.01$, and $s_\beta^d = 0.7056$, lead to the down quark masses: $m_d = 5.4663$ MeV, $m_s = 107.699$ MeV, $m_b = 4.216$ GeV, $M_D = 500.008$ GeV, and the mixing matrix

$$V_{dL}^{(1)} = \begin{pmatrix} 0.61120 & -0.79139 & -0.01093 & 9.2 \times 10^{-5} \\ 0.79127 & 0.61129 & -0.01429 & 1.2 \times 10^{-4} \\ 0.01799 & 8.04 \times 10^{-5} & 0.99983 & 0.00152 \\ -1.78 \times 10^{-4} & -7.96 \times 10^{-7} & -0.00152 & 0.99999 \end{pmatrix}. \quad (36)$$

B. Sector u:

Parameter space: $(\sqrt{\lambda_-})_u = 358.2$ GeV, $(\sqrt{\lambda_+})_u = 1241.44$ TeV, $r_u = .04$, $(\eta_+/a_3b_3)_u = -3.20432$, $(\eta_-/\eta_+)_u = 0$, $s_\alpha^u = .01$ and $s_\beta^u = 0.02884$ yield the up quark masses $m_u = 2.4$ MeV, $m_c = 1.2$ GeV, $m_t = 172$ GeV, $M_U = 1241.44$ TeV, and the mixing

$$V_{uL}^{(1)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.99900 & 0.04458 & -1.80 \times 10^{-7} \\ 0 & -0.04458 & 0.99900 & 4.34 \times 10^{-6} \\ 0 & 3.7366 \times 10^{-7} & -4.33 \times 10^{-6} & 1 \end{pmatrix}. \quad (37)$$

The See-saw V^o contribution, Eq.(31) with $s_o = -0.417698$, Eq.(34) reads

$$V^o = \begin{pmatrix} 0.41713 & 0.90858 & -0.02168 & -2.16 \times 10^{-4} \\ -0.90871 & 0.41736 & 0.00723 & 7.23 \times 10^{-5} \\ 0.01562 & 0.01669 & 0.99963 & 0.01 \\ 1.562 \times 10^{-4} & 1.66 \times 10^{-4} & 0.0100 & 0.0001 \end{pmatrix} \quad (38)$$

C. $(V_{CKM})_{4 \times 4}$

The above up and down quark mixing matrices $V_{uL}^{(1)}$, $V_{dL}^{(1)}$ and V^o yield the quark mixing matrix

$$(V_{CKM})_{4 \times 4} = \begin{pmatrix} 0.97428 & 0.22530 & 0.00413 & -3.97 \times 10^{-4} \\ -0.22527 & 0.97341 & 0.04133 & -3.96 \times 10^{-4} \\ 0.00528 & -0.04120 & 0.99902 & -0.01151 \\ 4.77 \times 10^{-5} & -2.25 \times 10^{-5} & -0.0100 & 1.15 \times 10^{-4} \end{pmatrix} \quad (39)$$

Notice that the $(V_{CKM})_{3 \times 3}$ sub-matrix is nearly a unitary mixing matrix, which is consistent with the allowed measured values for quark mixing reported in the PDG [16].

D. Charged Leptons:

For this sector, the parameter space: $(\sqrt{\lambda_-})_e = 9.14301$ GeV, $(\sqrt{\lambda_+})_e = 23816.4$ TeV, $r_e = 0.05$, $(\eta_+/a_3 b_3)_e = -1.99484$, $(\eta_-/\eta_+)_e = 0$, $s_\alpha^e = 0.001$ and $s_\beta^e = 0.00038$, reproduce the known charged lepton masses: $m_e = 0.511$ MeV, $m_\mu = 105.658$ MeV, $m_\tau = 1776.82$ MeV and $M_E \approx 23816.4$ TeV

E. Neutrinos 3+1:

For this sector, the parameter space: $(\sqrt{\lambda_-})_\nu = 0.048$ eV, $(\sqrt{\lambda_+})_\nu = 0.54$ eV, $r_\nu = 0.04$, $(\eta_+/a_3 b_3)_e = 0.01$, $(\eta_-/\eta_+)_e = 4.7$, $s_\alpha^\nu = 0.2$ and $s_\beta^\nu = 0.3992$, fit the neutrinos masses

$$(m_1, m_2, m_3, m_4) = (0.0102, 0.0134, 0.0511, 0.5398) \text{ eV}, \quad (40)$$

the squared mass differences

$$m_2^2 - m_1^2 \approx 7.6 \times 10^{-5} \text{ eV}^2, \quad m_3^2 - m_2^2 \approx 2.43 \times 10^{-3} \text{ eV}^2$$

$$m_4^2 - m_1^2 \approx 0.29 \text{ eV}^2, \quad (41)$$

and for charged leptons and neutrinos in version I, the first row of lepton mixing angles

$$(U_{PMNS})_{11} = 0.8145, \quad (U_{PMNS})_{12} = 0.5773$$

$$(U_{PMNS})_{13} = 0.0422, \quad (U_{PMNS})_{14} = 1.27 \times 10^{-4} \quad (42)$$

F. FCNC's in $K^o - \bar{K}^o$ meson mixing

The $SU(3)$ horizontal gauge bosons contribute to new FCNC's, in particular they mediate $\Delta F = 2$ processes at tree level. Here we compute their leading contribution to $K^o - \bar{K}^o$ meson mixing. In the previous scenario the up quark sector does not contribute to $(V_{CKM})_{12}$, and hence the effective hamiltonian from the tree level diagrams, Fig.2, mediated by the $SU(2)$ horizontal gauge bosons of mass M_1 to the $\mathcal{O}_{LL}(\Delta S = 2) = (\bar{d}_L \gamma_\mu s_L)(\bar{d}_L \gamma^\mu s_L)$ operator, is given by

$$\mathcal{H}_{eff} = C_{\bar{d}s} \mathcal{O}_{LL} \quad , \quad C_{\bar{d}s} \approx \frac{g_H^2}{4} \frac{1}{M_1^2} \frac{r_d^4}{(1 + r_d^2)^2} (s_{12}^d)^2 \quad , \quad (43)$$

and then contribute to the $K^o - \bar{K}^o$ mass difference as

$$\Delta M_K \approx \frac{2\pi^2}{3} \frac{\alpha_H}{\pi} \frac{r_d^4}{(1 + r_d^2)^2} (s_{12}^d)^2 \frac{F_K^2}{M_1^2} B_K(\mu) M_K \quad . \quad (44)$$

It is worth to point out the double mixing suppression in ΔM_K , Eq.(44); one from the see-saw mechanism due to the $r_d = (\frac{a'}{a_3})_d$ parameter, and the one from d-quark mixing s_{12}^d . Using the input values: $r_d = 0.052$, $\frac{\alpha_H}{\pi} = 0.2$, $s_{12}^d = 0.79139$, $F_K = 160$ MeV, $M_K = 497.614$ MeV and $B_K = 0.8$, one gets

$$\Delta m_K \approx 2.77 \times 10^{-12} \text{ MeV} \quad , \quad (45)$$

which is lower than the current experimental bound[16], $(\Delta m_K)_{Exp} = M_{K_L} - M_{K_S} \approx 3.48 \times 10^{-12}$ MeV. The quark mixing alignment in Eqs.(36 - 39) avoids tree level contributions to $D^0 - \bar{D}^0$ mixing mediated by the $SU(2)$ horizontal gauge bosons.

VI. CONCLUSIONS

We have reported a low energy parameter space within a $SU(3)$ flavor symmetry model extension, which combines tree level "Dirac See-saw" mechanisms and radiative corrections to implement a successful hierarchical spectrum for charged fermion masses and quark mixing. In section 5 we have reported the predicted values for quark and charged lepton masses and quark mixing matrix $(V_{CKM})_{4 \times 4}$ within allowed experimental values reported in PDG

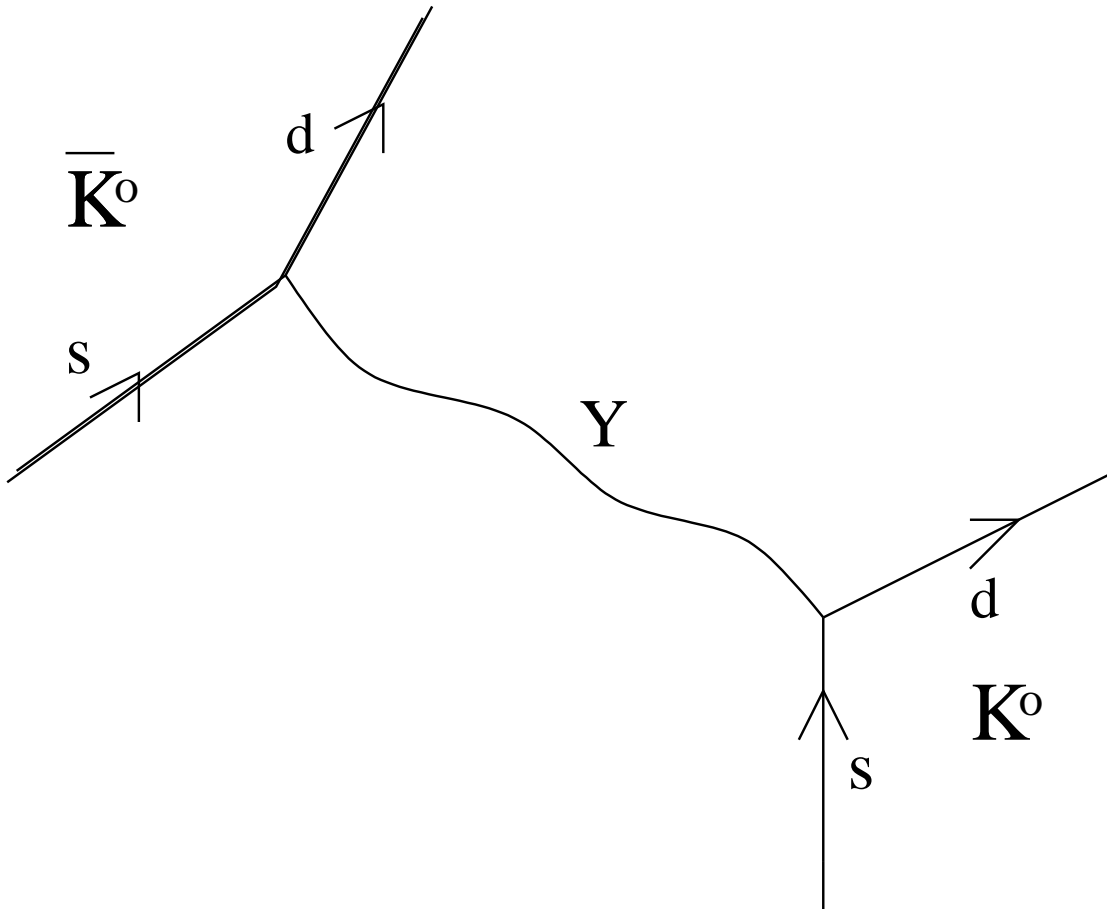


FIG. 2: Tree level contribution to $K^0 - \bar{K}^0$ from the light $SU(2)$ horizontal gauge bosons.

2010, coming from an input space parameter region with the lower horizontal scale $M_1 = 4$ TeV and a D vector-like quark mass of the order of 500 GeV. Furthermore, motivated by the recent LSND and MiniBooNe short-baseline neutrino oscillation experiments we are able to fit in the 3+1 scenario the neutrino squared mass differences $m_2^2 - m_1^2 \approx 7.6 \times 10^{-5} \text{ eV}^2$, $m_3^2 - m_2^2 \approx 2.43 \times 10^{-3} \text{ eV}^2$ and $m_4^2 - m_1^2 \approx 0.29 \text{ eV}^2$. Hence some of the new particles introduced in this model are within reach at the current LHC experiments, while simultaneously being consistent with present bounds on FCNC in $K^0 - \bar{K}^0$ and $D^0 - \bar{D}^0$ meson mixing.

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VIII. APPENDIX A:

DIAGONALIZATION OF THE GENERIC DIRAC SEE-SAW MASS MATRIX

$$\mathcal{M}^o = \begin{pmatrix} 0 & 0 & 0 & a_1 \\ 0 & 0 & 0 & a_2 \\ 0 & 0 & 0 & a_3 \\ b_1 & b_2 & b_3 & c \end{pmatrix} \quad (46)$$

Using a biunitary transformation $\psi_L^o = V_L^o \chi_L$ and $\psi_R^o = V_R^o \chi_R$ to diagonalize \mathcal{M}^o , the orthogonal matrices V_L^o and V_R^o may be written explicitly as the following two versions

Version I:

$$V_L^o = \begin{pmatrix} \frac{a_2}{a'} & \frac{a_1 a_3}{a' a} & \frac{a_1}{a} \cos \alpha & \frac{a_1}{a} \sin \alpha \\ -\frac{a_1}{a'} & \frac{a_2 a_3}{a' a} & \frac{a_2}{a} \cos \alpha & \frac{a_2}{a} \sin \alpha \\ 0 & -\frac{a'}{a} & \frac{a_3}{a} \cos \alpha & \frac{a_3}{a} \sin \alpha \\ 0 & 0 & -\sin \alpha & \cos \alpha \end{pmatrix} \quad (47)$$

$$V_R^o = \begin{pmatrix} \frac{b_2}{b'} & \frac{b_1 b_3}{b' b} & \frac{b_1}{b} \cos \beta & \frac{b_1}{b} \sin \beta \\ -\frac{b_1}{b'} & \frac{b_2 b_3}{b' b} & \frac{b_2}{b} \cos \beta & \frac{b_2}{b} \sin \beta \\ 0 & -\frac{b'}{b} & \frac{b_3}{b} \cos \beta & \frac{b_3}{b} \sin \beta \\ 0 & 0 & -\sin \beta & \cos \beta \end{pmatrix} \quad (48)$$

Version II:

$$V_L^o = \begin{pmatrix} \frac{a_1 a_3}{a' a} & -\frac{a_2}{a'} & \frac{a_1}{a} \cos \alpha & \frac{a_1}{a} \sin \alpha \\ \frac{a_2 a_3}{a' a} & \frac{a_1}{a'} & \frac{a_2}{a} \cos \alpha & \frac{a_2}{a} \sin \alpha \\ -\frac{a'}{a} & 0 & \frac{a_3}{a} \cos \alpha & \frac{a_3}{a} \sin \alpha \\ 0 & 0 & -\sin \alpha & \cos \alpha \end{pmatrix}, \quad (49)$$

$$V_R^o = \begin{pmatrix} \frac{b_1 b_3}{b' b} & -\frac{b_2}{b'} & \frac{b_1}{b} \cos \beta & \frac{b_1}{b} \sin \beta \\ \frac{b_2 b_3}{b' b} & \frac{b_1}{b'} & \frac{b_2}{b} \cos \beta & \frac{b_2}{b} \sin \beta \\ -\frac{b'}{b} & 0 & \frac{b_3}{b} \cos \beta & \frac{b_3}{b} \sin \beta \\ 0 & 0 & -\sin \beta & \cos \beta \end{pmatrix}, \quad (50)$$

$$\lambda_{\pm} = \frac{1}{2} \left(B \pm \sqrt{B^2 - 4D} \right) \quad (51)$$

are the nonzero eigenvalues of $\mathcal{M}^o \mathcal{M}^{oT}$ ($\mathcal{M}^{oT} \mathcal{M}^o$), with

$$B = a^2 + b^2 + c^2 = \lambda_- + \lambda_+ \quad , \quad D = a^2 b^2 = \lambda_- \lambda_+ , \quad (52)$$

$$\cos \alpha = \sqrt{\frac{\lambda_+ - a^2}{\lambda_+ - \lambda_-}} \quad , \quad \sin \alpha = \sqrt{\frac{a^2 - \lambda_-}{\lambda_+ - \lambda_-}} , \quad (53)$$

$$\cos \beta = \sqrt{\frac{\lambda_+ - b^2}{\lambda_+ - \lambda_-}} \quad , \quad \sin \beta = \sqrt{\frac{b^2 - \lambda_-}{\lambda_+ - \lambda_-}} .$$

$$\cos \alpha \cos \beta = \frac{c \sqrt{\lambda_+}}{\lambda_+ - \lambda_-} \quad , \quad \cos \alpha \sin \beta = \frac{b c^2 \sqrt{\lambda_+}}{(\lambda_+ - b^2)(\lambda_+ - \lambda_-)} \quad (54)$$

$$\sin \alpha \sin \beta = \frac{c \sqrt{\lambda_-}}{\lambda_+ - \lambda_-} \quad , \quad \sin \alpha \cos \beta = \frac{a c^2 \sqrt{\lambda_-}}{(\lambda_+ - a^2)(\lambda_+ - \lambda_-)}$$

Notice that in the space parameter $a^2 \ll c^2$, b^2 , $\frac{\lambda_-}{\lambda_+} \ll 1$, so that we may approach the eigenvalues as

$$\lambda_- \approx \frac{D}{B} \approx \frac{a^2 b^2}{c^2 + b^2} \quad , \quad \lambda_+ \approx c^2 + b^2 + a^2 - \frac{a^2 b^2}{c^2 + b^2} \quad (55)$$